Lecture 15 **2013/01/07**

Outline

1. **Motivation**
2. **Dynamical Systems and Global Existence**
3. **Limit Sets and Attractors**
4. **Poincare-Bendixson Theory in R^2**
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1. Motivation

- For a linear system with constant coefficients, it has an explicit trajectory (solution) structure. It is natural to expect to know this similar trajectory (solution) information of a nonlinear autonomous system. The corresponding theory is called the theory of a dynamical system, or the geometric (qualitative) theory of a dynamical system.

- Unlike its counterpart: linear systems, dynamical systems are much difficult. It has mainly two parts: a local theory and a global theory. We have learned the local theory like Existence and Uniqueness; Dependence on Initial Conditions and Parameters; Linearization; Stable Manifold Theorem; Hartman-Grobman Theorem, Central Manifold Theory etc. As for the global theory, we going to introduce some basic concepts and to know some about Poincare-Bendixson theory for planar systems, where the theory is relatively complete. For dynamical systems with $n \geq 3$, it is a challenging problem which is the subject of much mathematical research at this time.

2. Dynamical Systems and Global Existence

1) Dynamical Systems

Definition 15.1 A dynamical system on an open set D is a C^1-map : $\varphi : R \times D \to D$,

where $D \subseteq R^n$ and if $\varphi_t(x) = \varphi(t; x)$ then φ_t satisfies

(1) $\varphi_0(x) = x$ for all $x \in D$;

(2) $\varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x)$ for all $s, t \in R$ and $x \in D$,

where $\varphi_t \circ \varphi_s(x) \overset{def.}{=} \varphi_t(\varphi_s(x))$.

Remark 15.1 It follows from Definition 15.1 that for each $t \in R$, φ_t is a C^1-map of D into D which has a C^1-inverse, φ_{-t}. φ_t with $t \in R$ is a one-parameter family of diffeomorphisms on D.

Remark 15.2 In general, if $\varphi(t; x)$ is a dynamical system on $D \subseteq R^n$, then, the function

$$\frac{d}{dt}\varphi(t; x)|_{t=0} \overset{def.}{=} f(x)$$

defines a C^1-vector field on D and for each $x_0 \in D$, $\varphi(t; x_0)$ is the solution of IVP

$$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}. \tag{15.1}$$

Remark 15.3 However, for each $x_0 \in D$, the maximal interval of existence $I_{\max}(x_0)$ of $\varphi(t; x_0)$ may not equal to $R = (-\infty, \infty)$. Blow-up (escape at finite time) phenomenon should be eliminated.

Definition 15.2 Suppose that $f \in C^1(D_1)$ and $g \in C^1(D_2)$, where D_1 and D_2 are open subsets of R^n. Then, $x' = f(x)$ and $x' = g(x)$ are said to be **topologically equivalent** if there exists an orientation preserving homeomorphism $H : D_1 \to D_2$ which maps trajectories of $x' = f(x)$ onto trajectories of $x' = g(x)$. Moreover, if the homeomorphism H also preserves the parameterization by time, then, the two systems are said to be **topologically conjugate**.

2) Global Existence

Theorem 15.1 Suppose that $f \in C^1(R^n)$ in (15.1). Then, for each $x_0 \in R^n$, the IVP

$$\begin{cases} \dot{x} = \dfrac{f(x)}{1 + \| f(x) \|} \\ x(0) = x_0 \end{cases} \tag{15.2}$$

has a unique solution $x(t)$ defined for all $t \in R$, i.e. (15.2) defines a dynamical system on R^n. Moreover, (15.2) is topologically equivalent to (15.1) on R^n.

Proof. Clearly, we have $f \in C^1(R^n) \Leftrightarrow \frac{f}{1+\|f\|} \in C^1(R^n)$. Since $F(x) = \frac{f(x)}{1+\|f(x)\|}$ satisfies $\|F(x)\| \le 1$, the integral form of (15.2) is given by

$$x(t) = x_0 + \int_0^t F(s, x(s))\, ds .$$

Then, we have $\|x(t)\| \le \|x_0\| + |t|$, which implies $I_{\max}(x_0) = (\alpha, \beta)$ with $\alpha = -\infty$ and $\beta = \infty$ by contradiction based on Continuation Theorem.

As for the topological equivalence between (15.1) and (15.2), the homeomorphism H is simply the identity on R^n, and the time t along the solution $x(t)$ is rescaled according to the formula $\tau = \int_0^t (1+\|f(s)\| ds$. □

Remark 15.4 (15.1) and (15.2) is topologically equivalent. But it is not topologically conjugate in general.

Remark 15.5 If $f \in C^1(D)$, where D is a proper subset of R^n, the above normalization in Theorem 15.1 will not, in general, lead to a dynamical system as the next example shows.

Example 15.1 For $x_0 > 0$, the IVP $\begin{cases} \dot{x} = \frac{1}{2x} \\ x(0) = x_0 \end{cases}$ has the unique solution $x(t) = \sqrt{t + x_0^2}$ defined on $I_{\max}(x_0) = (-x_0^0, \infty)$. The function $f(x) = \frac{1}{2x} \in C^1(D)$, where $D = (0, \infty)$. The normalized IVP

$$\begin{cases} \dot{x} = \frac{1/2x}{1+1/2x} = \frac{1}{1+2x} \\ x(0) = x_0 \end{cases}$$

has the unique solution $x(t) = -\frac{1}{2} + \sqrt{t + (x_0 + \frac{1}{2})^2}$, defined on

$$I_{\max}(x_0) = (-(x_0 + \frac{1}{2})^2, \infty) \ne (-\infty, \infty).$$

Remark 15.6 However, a slightly more subtle rescaling of the time along trajectories of (15.1) does lead to a dynamical system, which is topologically equivalent to (15.1) even when D is a proper subset of R^n.

Theorem 15.2 Suppose that $f \in C^1(D)$ in (15.1), where D is a subset of R^n. Then, there exists a function $F \in C^1(D)$ such that

$$\dot{x} = F(x) \tag{15.3}$$

defines a dynamical system on D. Moreover, (15.3) is topologically equivalent to (15.1) on D.

Proof. Omitted. Consult “Differential Equations and Dynamical Systems” by Lawrence Perko, Theorem 2, Chapter 3, if needed.

Remark 15.7 From now on, without loss of generality, we may suppose that (15.1) defines a dynamical system on D, where D is a subset of R^n.

Two more global existence results which are of some interest are presented. Their proofs are left for students because they are simple.

Theorem 15.3 Suppose that $f \in C^1(R^n)$ and that $f(x)$ satisfies the global Lipschitz condition

$$\| f(x_1) - f(x_2) \| \le L \| x_1 - x_2 \|$$

for all $x_1, x_2 \in R^n$. Then, for each $x_0 \in R^n$, the IVP (15.1) has a unique solution $x(t)$ defined for all $t \in R$.

Theorem 15.4 Suppose that $f \in C^1(D)$, where D is a compact set of R^n. Then, for each $x_0 \in R^n$, the IVP (15.1) has a unique solution $x(t)$ defined for all $t \in R$.

3. Limit Sets and Attractors

1) Limit Sets

Definition 15.2 Let $x(t; x_0)$ be a solution of the system (15.2) for $t \in R$. A point $p \in D$ is an ω**-limit point** of $x(t; x_0)$ ($\gamma(x_0)$) if there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n; x_0) = p$. The ω**-limit set** of $x(t; x_0)$ ($\gamma(x_0)$) is denoted

$\Omega^+(x_0)$. A point $p \in D$ is an α **-limit point** of $x(t; x_0)$ ($\gamma(x_0)$) if there exists $\{t_n\}$ with $t_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n; x_0) = p$. The α **-limit set** of $x(t; x_0)$ ($\gamma(x_0)$) is denoted $\Omega^-(x_0)$.

Example 15.2 If $x(t; x_0) \equiv x_0$, then $\Omega^+(x_0) = \Omega^-(x_0) = \{x_0\}$. That is, the limit set of equilibrium is itself.

If $x(t; x_0)$ is a periodic orbit, then $\Omega^+(x_0) = \Omega^-(x_0) = x(t; x_0)$. That is, the limit set of a periodic orbit is also itself.

If an equilibrium $x = x_0$ is AS (unstable), then it is ω (α)-limit set of its nearby trajectories; If $x(t; x_0)$ is stable (unstable) limit cycle, then $x(t; x_0)$ is ω (α) -limit set of its nearby trajectories.

Theorem 15.4 (i.e. Lemma 13.4) If $x = x^*$ is an ω-limit point of $x(t; x_0)$, then any point on the trajectory $x(t; x^*)$ is also a ω-limit point of $x(t; x_0)$.

Proof. Since $x = x^*$ is an ω -limit point, there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ s. t. $\lim_{n\to\infty} x(t_n; x_0) = x^*$ by definition. Suppose that $x(\tau; x^*)$ is any point of $x(t; x^*)$. By Lemma 13.3, we have

$$x(t_n + \tau; x_0) = x(\tau; x(t_n; x_0)).$$

Then

$$\lim_{n\to\infty} x(\tau + t_n;\ x_0) = \lim_{n\to\infty} x(\tau; x(t_n; x_0)) = x(\tau; \lim_{n\to\infty} x(t_n; x_0)) = x(\tau; x^*).$$

This shows that $x(\tau; x^*)$ is also a ω-limit point. □

Remark 15.6 Theorem 15.4 shows that $\Omega^+(x_0)$ consists of whole trajectories of (13.2). This is a very important property for the autonomous system.

Theorem 15.5 The ω (α)-limit set $\Omega^+(x_0)$ ($\Omega^-(x_0)$) of a trajectory $\gamma(x_0)$ is closed subset of D and if $\gamma(x_0)$ is contained in a compact subset of R^n, then $\Omega^+(x_0)$ ($\Omega^-(x_0)$) is non-empty, compact, invariant, and connected subset of D, Moreover, $\varphi(t; x_0) \to \Omega^+(x_0)$ as $t \to \infty$ ($\varphi(t; x_0) \to \Omega^-(x_0)$ as $t \to -\infty$).

Proof. First, it follows from Definition 15.2 that $\Omega^+(x_0) \subset D$. To show that $\Omega^+(x_0)$ is closed subset of D, let $\{p_n\} \in \Omega^+(x_0)$ such that $\lim_{n\to\infty} p_n = p$ and show that $p \in \Omega^+(x_0)$. Since $p_n \in \Omega^+(x_0)$, then, for each $n \ge 1$, there exists a sequence $t_k^{(n)} \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \varphi(t_k^{(n)}; x_0) = p_n .$$

Moreover, we may assume that $t_k^{(n+1)} > t_k^{(n)}$ since otherwise we can choose a subsequence of $t_k^{(n)}$ with this property. $\lim_{k\to\infty} \varphi(t_k^{(n)}; x_0) = p_n$ implies that for each n, there exists sufficiently large $T_n > 0$ such that for $k > T_n$, we have

$$\| \varphi(t_k^{(n)}; x_0) - p_n \| < \frac{1}{n} .$$

Let $t_n = t_{T_n}^{(n)}$. Then $t_n \to \infty$ and by the triangle inequality,

$$\| \varphi(t_n; x_0) - p \| \le \| \varphi(t_n; x_0) - p_n \| + \| p_n - p \| \le \frac{1}{n} + \| p_n - p \| \to \infty$$

as $n \to \infty$. Thus $p \in \Omega^+(x_0)$. Hence, $\Omega^+(x_0)$ is closed subset of D.

If $\gamma(x_0) \subset K$, where K is a compact subset of R^n, and for any $p \in \Omega^+(x_0)$, there exists $\varphi(t_n; x_0) \to p$ as $n \to \infty$, then $p \in K$ since $\varphi(t_n; x_0) \subset K$ and K is a compact. Thus, $\Omega^+(x_0) \subset K$. Therefore, $\Omega^+(x_0)$ is compact since a closed subset of a compact set is compact.

Moreover, $\Omega^+(x_0) \ne 0$ since any bounded sequence contains a convergent subsequence which converges to a point in $\Omega^+(x_0) \subset K$ by the Bolzano-Weierstrass theorem.

Next for showing the invariance of $\Omega^+(x_0)$. Let $p \in \Omega^+(x_0)$ and show that $\varphi(t; p) \in \Omega^+(x_0)$ for all $t \ge 0$. Since $p \in \Omega^+(x_0)$, there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t_n; x_0) \to p$ as $n \to \infty$. By the group property,

$$\varphi(t+t_n;\, x_0) = \varphi(t;\, \varphi(t_n;\, x_0)),$$

where, for sufficiently large $n > 0$, $t + t_n \ge 0$. By continuity,

$$\lim_{n\to\infty} \varphi(t+t_n;\, x_0) = \lim_{n\to\infty} \varphi(t;\, \varphi(t_n;\, x_0)) = \varphi(t;\, p),$$

which shows that $\varphi(t;\, p) \in \Omega^+(x_0)$ for all $t \ge 0$. Therefore, $\Omega^+(x_0)$ is invariant with respect to the flow φ_t of (15.1).

Suppose that $\Omega^+(x_0)$ is not connected. Then, there exist two non-empty, disjoint, closed sets A and B such that $\Omega^+(x_0) = A \cup B$.

Let $d = \inf_{x\in A,\, y\in B} \| x - y \| > 0$. Since the points of A and B are ω-limit point of $\Omega^+(x_0)$, for $d > 0$, there exists arbitrarily large $t > 0$ such that

$$d(\varphi(t;\, x_0),\, A) < \frac{d}{2} \quad \text{and} \quad d(\varphi(t;\, x_0),\, B) < \frac{d}{2}.$$

Let $g(t) = d(\varphi(t;\, x_0),\, A)$. $g(t)$ is a continuous function. Then, there exists $t_n \to \infty$ such that

$$d(\varphi(t_n;\, x_0),\, A) = \frac{d}{2} \quad \text{for all} \quad n \ge 0.$$

Since $\varphi(t_n;\, x_0) \subset K$, there exists a subsequence converging to $p \in \Omega^+(x_0)$. So it follows that $d(p,\, A) = \frac{d}{2}$. But

$$d(p,\, B) \ge d(A,\, B) - d(p,\, A) = d - \frac{d}{2} = \frac{d}{2},$$

which implies that $p \notin A$ and $p \notin B$. Then, $p \notin \Omega^+(x_0)$. This is a contradiction. Thus, $\Omega^+(x_0)$ is connected. Similar to show that $\Omega^-(x_0)$ is connected.

Finally, suppose that $\varphi(t;\, x_0) \to \Omega^+(x_0)$ as $t \to \infty$ is not true. Then, there exists $\varepsilon > 0$ and $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$d(\varphi(t_n;\, x_0),\, \Omega^+(x_0)) > \varepsilon.$$

Since $\varphi(t_n;\, x_0) \subset K$, there exists a convergent subsequence $\varphi(t_{n_k};\, x_0) \to p$ as $k \to \infty$. Then, $p \in \Omega^+(x_0)$ and at the same time, it keeps that $d(p,\, \Omega^+(x_0)) > \varepsilon$.

This is a contradiction. It is similar to show for $\Omega^-(x_0)$. □

Remark 15.7 This result is extremely important both in dynamical systems and Lyapunov stability theory.

Theorem 15.6 If $x = p$ is an ω-limit point of $\varphi(t; x_0)$, then any point on the trajectory $\varphi(t; p)$ is also a ω-limit point of $\varphi(t; x_0)$.

Remark 15.8 Theorem 15.6 is just Lemma 13.4. It shows that for all points $p \in \Omega^+(x_0)$, $\Rightarrow$ $\varphi(t; p) \in \Omega^+(x_0)$ for all $t \in R$. i.e. $\varphi_t(\Omega^+(x_0)) \subset \Omega^+(x_0)$.

2) Attractors

Definition 15.3 A closed invariant set $A \subset D$ is called **an attracting set** of (15.1) if there exists some neighborhood U of A such that for all $x \in U$, $\varphi_t(x) \in U$ for all $t \geq 0$ and $\varphi_t(x) \to A$ as $t \to \infty$. **An attractor** of (15.1) is an attracting set which contains a dense orbit.

Example 15.3 Consider the system

$$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases}.$$

In polar coordinates, we have

$$\begin{cases} r' = r(1 - r^2) \\ \theta' = 1 \end{cases},$$

where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\dfrac{y}{x} \end{cases}$. Since

$$xx' + yy' = (x^2 + y^2)[1 - (x^2 + y^2)]; \quad xy' - yx' = -(x^2 + y^2),$$

it follows that

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{1}{\sqrt{x^2 + y^2}}(xx' + yy') = \dfrac{1}{\sqrt{x^2 + y^2}}(x^2 + y^2)[1 - (x^2 + y^2)] = r(1 - r^2) \\ \dfrac{d\theta}{dt} = \dfrac{1}{x^2 + y^2}(xy' - yx') = \dfrac{1}{x^2 + y^2}[-(x^2 + y^2)] = -1 \end{cases}.$$

$$\Rightarrow \quad r=0; \quad r=1.$$

$$r=0 \quad \Leftrightarrow \quad \text{the equilibrium: } (0,0);$$

$$r=1 \quad \Leftrightarrow \quad \text{the closed orbit: } x^2+y^2=1.$$

Discussion:

(1) If $0<r<1$, $\Rightarrow$ $\dfrac{dr}{dt}>0$ $\Rightarrow$ $r(t)$ is increased in $t\geq 0$;

(2) If $r>1$, $\Rightarrow$ $\dfrac{dr}{dt}<0$ $\Rightarrow$ $r(t)$ is decreased in $t\geq 0$;

(3) $\dfrac{d\theta}{dt}=-1<0$ $\Rightarrow$ $\theta(t)$ is decreased $\Rightarrow$ the counter-clockwise direction for trajectories.

Conclusion;

Γ_0: $r=1$ is a stable limit cycle. See Fig. 15.1.

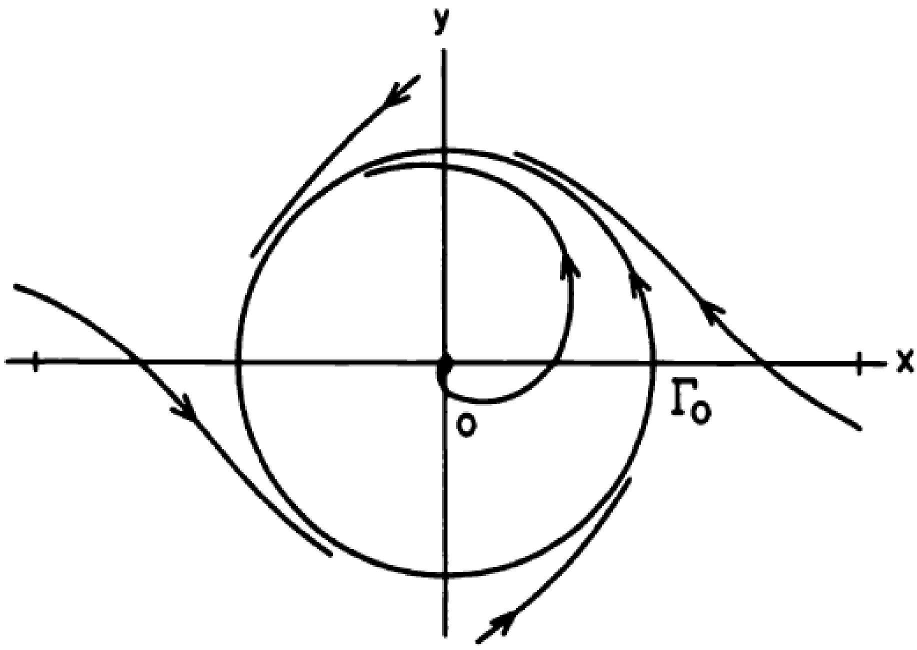

Fig. 15.1 A stable limit cycle Γ_0 which is an attractor of the system

3) LaShalle's Invariant Principle

Theorem 15.7 Let $K\subset D$ be a compact, positively invariant. Let $V:D\to R$ be of C^1 such that $\dot{V}(x)\leq 0$ in K. Let $S=\{x\in K \mid \dot{V}(x)=0\}$. Let M be the largest invariant set in S. Then every solution of (15.1) starting in K approaches M as $t\to\infty$.

Proof. Let $\varphi(t;x_0)$ be a solution of (15.1), $x_0\in K$ and $\Omega^+(x_0)$ be a positively limit set.

Step 1. To show that $\lim_{t\to\infty} V(\varphi(t;x_0))$ exists. Since $\dot{V}(x)\le 0$ in K, $V(\varphi(t;x_0))$ is decreasing in t. Since $V(x)$ is continuous on a compact set K, it is bounded from below on K. Therefore, $\lim_{t\to\infty} V(\varphi(t;x_0)) = a$ exists.

Step 2. If we can show that $\Omega^+(x_0)$ is invariant in S $\Rightarrow$ $\Omega^+(x_0)\subset M$ since M is the largest invariant set in E, and then,

$$\Rightarrow \quad \Omega^+(x_0)\subset M \subset S \subset K .$$

(1) Since $\Omega^+(x_0)$ is invariant in K, so we only need to show that $\Omega^+(x_0)\subset S$.

If $\varphi(t;x_0)\in S$ for all $t\ge 0$, then for any $\{t_n\}\ge 0$ with $\lim_{n\to\infty} t_n = \infty$, $\varphi(t_n;x_0)\in S$ $\Leftrightarrow$ $\varphi(t_n;x_0)\subset K$ and $\dot{V}(\varphi(t_n;x_0))\equiv 0$. Note that $\dot{V}$ is continuous and K is compact, it implies that

$$p = \lim_{n\to\infty}\varphi(t_n;x_0)\in K \text{ and } \dot{V}(\lim_{n\to\infty}\varphi(t_n,x_0))\equiv \dot{V}(p)=0$$

$$\Rightarrow \quad p\in S \quad \Rightarrow \quad \Omega^+(x_0)\subset S .$$

(2) It shows that $\dot{V}(\varphi(t;x_0))\equiv 0$ for all $t\ge 0$ for $\varphi(t;x_0)\in S$.

For any $p\in\Omega^+(x_0)$, there exists $\{t_n\}\ge 0$ with $\lim_{n\to\infty} t_n=\infty$ such that $\lim_{n\to\infty}\varphi(t_n;x_0)=p$. It implies that $V(p)=\lim_{n\to\infty}V(\varphi(t_n;x_0))=a$, $\Rightarrow$ $V(x)\equiv a$ on $\Omega^+(x_0)$. Since $\varphi(t;x_0)\subset\Omega^+(x_0)$ for all $t\ge 0$ (Theorem 15.5) $\Rightarrow$ $V(\varphi(t;x_0))\equiv a$ for all $t\ge 0$. $\Rightarrow$ $\dot{V}(\varphi(t;x_0))\equiv 0$ for all $t\ge 0$ $\Rightarrow$ $\varphi(t;x_0)\subset S$. Therefore, $\Omega^+(x_0)$ is invariant in S.

Since $\Omega^+(x_0)\subset M$, $\varphi(t;x_0)\to\Omega^+(x_0)$ as $t\to\infty$ by Theorem 15.5, Hence, $\varphi(t;x_0)\to M$ as $t\to\infty$. □

Remark 15.9

1) If $M=\{0\}$, then $\varphi(t;x_0)\to 0$ as $t\to\infty$ $\Rightarrow$ $x=0$ is AS if it is stable;

2) The key of the proof is to use the invariance of $\Omega^+(x_0)$ to locate its position

$$\Omega^+(x_0) \subset M \subset S \subset K ;$$

3) In practice, it is difficult to construct K, which is compact and positively invariant, so we often use the bounded $\Omega_\rho = \{x \mid V(x) \le \rho\}$ to replace K while to find $V(x) > 0$ although to find K does not have to be tied in with $V(x)$;

4) $V(x)$ does not have to be positively definite.

If one more condition is imposed on S, i.e. "no solution can stay identically in S, other than the equilibrium", Then, we obtain the following two corollaries immediately.

Corollary 15.1 (Theorem 13.1) Let $V : D \to R$ be C^1, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\};$$

$$\dot{V}(x) \le 0 \text{ in } D.$$

Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$. If there is no trajectory can stay identically in S, other than the origin. Then, the origin of (15.1) is AS.

Corollary 15.2 (Theorem 13.2) Let $V : D \to R$ be C^1, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\};$$

$$\dot{V}(x) \le 0 \text{ in } D;$$

$$\| x \| \to \infty \ \Rightarrow \ V(x) \to \infty.$$

Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$. If there is no trajectory can stay identically in S, other than the origin. Then, the origin of (15.1) is GAS.

4. Poincare-Bendixson Theory in R^2

1) Periodic Orbits and Limit Cycles

Definition 15.3 A cycle or **periodic orbit** of (15.1) is any closed solution curve of (15.1) which is not an equilibrium point of (15.1).

A periodic orbit γ is called **stable** if for each $\varepsilon > 0$ there is a neighborhood

U of γ such that for all $x \in U$, $d(\varphi(t;x), \gamma) < \varepsilon$ for all $t \geq 0$. γ is called **unstable** if it is not stable; and γ is called AS if for all $x \in U$ of γ,

$$\lim_{t \to \infty} d(\varphi(t;x), \gamma) = 0.$$

Remark 15.10 The stability of γ is an orbit stability. It is different from Lyapunov stability.

Definition 15.4 A limit cycle of (15.1) in R^2 is a cycle of (15.1) which is the ω or α-limit set of some trajectory of (15.1) other than γ**.**

If a cycle γ is the ω-limit set of every trajectory in some neighborhood of γ, then, γ is called **a stable limit cycle;** if a cycle γ is the α-limit set of every trajectory in some neighborhood of γ, then, γ is called **a unstable limit cycle;** and if γ is the ω-limit set of one trajectory other than γ and the α-limit set of another trajectory other than γ, then γ is called **a semi-stable limit cycle**.

Remark 15.11 A stable limit cycle is actually an AS cycle in the sense of Definition 15.3 and any stable limit cycle is an attractor.

Remark 15.12 The existence of a limit cycle and its stability are research concerns. They are difficult. How many limit cycles in all for (15.1) is a world open question, even in R^2. However, according to Jordan closed curve theorem, closed orbits separate R^2 into 2 connected components, the interior of and the exterior of the orbits. This makes the 2-dimensional case special and much more tractable than the general case.

2) Poincare-Bendixson Theory in R^2

Theorem 15.8 (Poincare-Bendixson Theorem) Let $f: U \to R^2$ be of C^1, where $U \subset R^2$ is an open set. Suppose that the positive trajectory γ with γ^+ of (15.1) is bounded. Then, if the limit set $\omega(\gamma)$ contains no equilibrium of (15.1), $\omega(\gamma)$ is a periodic orbit (limit cycle).

Remark 15.13 The proof of Theorem 15.8 is complicated and tedious. So it is omitted because of time limitation.

Corollary 15.3 Let K be a compact set which is positively invariant. Then, K contains either a limit cycle or an equilibrium point.

Proof. If K is positively invariant, then any trajectory starting in K is bounded because K is compact. The corollary follows from Theorem 15.8. □

Corollary 15.4 Let C be a periodic orbit and O be the open region in the interior of C. Then O contains either a limit cycle or an equilibrium point.

Proof. Let $D = O \cup C$. Then D is positively or negatively invariant. If O contains no limit cycle or equilibrium point, then for all $x \in D \subset R^2$, it must have by Theorem 15.8

$$\omega(\gamma(x)) = \alpha(\gamma(x)) = C .$$

So the trajectory spirals toward C both for positive and negative times. This is a contradiction. □

Remark 15.14

Theorem 15.9 (Bendixson Criterion) Let $f \in C^1(D)$, where $D \subset R^2$ is a simply connected region. If the divergence of the vector field f, ∇f is not identically zero and does not change sign in D, then (15.1) in R^2 has no closed orbit lying entirely in D.

Proof. Suppose that $C: \ x = \varphi(t), \ 0 \le t \le T$, is a closed orbit of (15.1) lying entirely in D. If S denotes the interior of C, it follows from Green's Theorem that

$$\iint_S \nabla f \, dx_1 dx_2 = \int_C f \cdot ds = \int_C (f_1 \, dx_2 - f_2 \, dx_1) = \int_C (f_1 \, f_2 - f_2 \, f_1) dt \equiv 0 .$$

And if ∇f is not identically zero and does not change sign in S, then it follows from the continuity of ∇f in S that the above double integral is either positive or negative. In either case this leads to a contradiction. Therefore, there is no closed orbit of (15.1) lying entirely in D. □

3) Illustrative Examples

Example 15.3 Consider

$$\begin{cases} x' = x - y - x^3 \\ y' = x + y - y^3 \end{cases}.$$

Let us take $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Along trajectories of the system, we have

$$V'(x, y) = x^2 + y^2 - x^4 - y^4.$$

If $x^2 + y^2 < 1$, $V'(x, y) = x^2 + y^2 - x^4 - y^4 > 0$;

If $x^2 + y^2 > 2$, $V'(x, y) < x^2 + y^2 + 2x^2y^2 - (x^2 + y^2)^2 \le 2(x^2 + y^2) - (x^2 + y^2)^2$

$$= (x^2 + y^2)(2 - (x^2 + y^2)) < 0.$$

This implies that the annular region $A = \{1 \le x^2 + y^2 \le 2\}$ is positively invariant. Any trajectory starting in A stays in A forever. There is only one equilibrium point is the origin, which is not in A. Poincare-Bendixson Theorem (in fact, Corollary 15.4) implies the existence of a periodic orbit contained in A. No uniqueness can be concluded in this time by Poincare-Bendixson Theorem. We need additional information for it.

Example 15.4 Consider

$$\begin{cases} x' = y + x(1 - x^2 - y^2) \\ y' = -x + y(1 - x^2 - y^2) \end{cases}.$$

Let us take $V(x, y) = x^2 + y^2$. Along trajectories of the system, we have

$$V'(x, y) = 2xy' + 2yx' = 2xy + 2x^2(1 - x^2 - y^2) - 2xy + 2y^2(1 - x^2 - y^2)$$

$$= 2V(x, y)(1 - V(x, y))$$

$V'(x, y) > 0$ for $V(x, y) < 1$ and $V'(x, y) < 0$ for $V(x, y) > 1$. Hence, on the level surface $V(x, y) = r_1$ with $0 < r_1 < 1$ all orbits move outward, while on the level surface $V(x, y) = r_2$ with $r_2 > 1$ all orbits move inward. This shows that the annular region

$$A = \{x \in R^2 \mid r_1 \le V(x, y) \le r_2\}$$

is positively invariant, closed, bounded, and free of equilibrium. The application of Poincare-Bendixson Theorem concludes that there is a periodic orbit in A. Since it is valid for any $r_1 < 1$ and $r_2 > 1$ closing to $r = 1$, we can make both r_1 and r_2

approach 1 so that the annular region A shrinks toward the unit circle.

We can arrive at the same conclusion by representing the system in the polar coordinates

$$x = r\cos\theta\,, \quad y = r\sin\theta$$

which yields

$$\dot{r} = r(1-r^2)\,, \quad \dot{\theta} = 1.$$

This shows not only that the unit circle is a periodic orbit, but also that it is the unique periodic one and that all trajectories (orbits), other than the origin, spiral toward the unit circle from inside or outside. The uniqueness of the periodic orbit could not be obtained by application of Poincare-Bendixson Theorem. This example is just an exception.

Example 15.5 Consider the Lienard equation given by

$$x'' + f(x)x' + g(x) = 0\,,$$

where $f(x)$ and $g(x)$ are Lipschitz continuous. If $h(x,y) = (y, -f(x)y - g(x))^T$ satisfies

$$\operatorname{div} h(x,y) = \frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} = -f(x)\,,$$

where $h(x,y) = (h_1(x,y), h_2(x,y))^T$. If $f(x)$ is either positive or negative, then Theorem 15.9 (Bendixson Criterion) implies that there is no periodic orbit.

5. Summary

- Global properties of a dynamical system are a main direction of research. In R^2, Poincare-Bendixson Theory is a celebrating result for the global properties. However, in R^n, $n \geq 3$, it is much more difficult than that of R^2. Lots of questions remain open.

- How to construct a proper annular region to conclude the existence of a periodic cycle is not easy work in practice by applying Poincare-Bendixson Theorem.

- Limit Sets and Attractors are important topics in qualitative analysis of a dynamical system.